数理解析研究所講究録
604巻 1987年 88-116


# OPTIMAL CONTROL OF LIQUID-SOLID PHASE TRANSITIONS

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## 0. INTRODUCTION

Physical phenomena of freezing and melting are elementary components of numerous complex processes like crystal growth, alloy solidification, ground consolidation, etc. Their analogues are faced in diffusion processes (possibly coupled with chemical reactions), in electrochemistry, in ecological processes of population dynamics, to list a few. We shall jointly classify such phenomena as liquid-solid phase transitions. All these processes share the feature of a discontinuous transformation from an ordered phase to disordered (in a certain sense). This is the behaviour typical for solid and liquid phases of any conventional material.

For the class of processes under consideration, boundary conditions which in general are time-dependent play a substantial role. Often, either those conditions or the relevant terms representing distributed sources may be treated as controlling factors. This gives rise to formulating suitable control problems, alternatively including cost functionals to be minimized or having closed-loop feedback structures (cf., [6,10] ). The first category refers to simple technological and economic criteria, the other reflects an attempt to stabilize the process along prescribed trajectory.

In this paper we develop an analysis of a certain class of standard boundary control problems for general two-phase Stefan-type processes. The Stefan problems can be considered as

*) Partially supported by the Faculty of Education, Chiba University and Research Program RP.I.02 of the MNSzWiT, Warsaw.

mathematical models of simple liquid-solid phase transitions (cf., [5,12] ). We use a variational inequality formulation of the models (cf., [9]). By exploiting diverse regularization techniques, we give a characterization of the relevant optimal solutions in the form of explicit optimality conditions. The characterization is constructive, it may be used for developing computational schemes (cf., [10]).

For other results on optimal control of various Stefan-type problems we refer to [1-3,8,13,14].

## 1. CONTROL PROBLEM STATEMENT

Let $\Omega$ be an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, with boundary $\Gamma$ regular enough. For $0 < T < \infty$, denote $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$.

To be specific, we shall consider the following boundary control problem:

$$\text{Minimize} \quad J(\vartheta,u) = \frac{1}{2} \|\vartheta - \vartheta_d\|^2_{L^2(Q)} + \frac{\alpha}{2} \|u\|^2_{\mathcal{U}} , \tag{1.1}$$

over the set of state functions $\vartheta \in L^2(Q)$ and controls $u \in \mathcal{U}$, subject to the state equations (two-phase Stefan problem)

$$\begin{cases} w' - \Delta\vartheta = \lambda , \quad w \in \gamma_o(\vartheta) \quad \text{in } Q , & (1.2a) \\ \partial_\nu \vartheta + p\,\vartheta = u \quad \text{on } \Sigma , & (1.2b) \\ \vartheta(0) = \vartheta_o , \quad w(0) = w_o \in \gamma_o(\vartheta_o) \quad \text{in } \Omega . & (1.2c) \end{cases}$$

Here $w' = \frac{\partial w}{\partial t}$, $\partial_\nu$ denotes the outward normal derivative on $\Gamma$; $\vartheta_d = \vartheta_d(x,t)$ , $\lambda = \lambda(x,t)$ , $\vartheta_o = \vartheta_o(x)$ , $w_o = w_o(x)$ , $p = p(x) \geq 0$ are given functions, $\alpha$ is a positive constant; $\gamma_o$ is a monotone graph (multivalued) in $\mathbb{R} \times \mathbb{R}$ . The control space $\mathcal{U}$ is alternatively assumed as either $\mathcal{U} = \mathcal{U}^o \equiv L^2(\Sigma)$ or $\mathcal{U} = \mathcal{U}^1 \equiv H^1(0,T;L^2(\Gamma))$ , equipped with the standard norms (up to an equivalence) $\|u\|_{U^0} = \|u\|_{L^2(\Sigma)}$ , $\|u\|_{U^1} = \|u(0)\|_{L^2(\Gamma)} + \|u'\|_{L^2(\Sigma)}$ .

(1.2) is referred to as the enthalpy fixed-domain formulation of two-phase Stefan problem, with the enthalpy graph $\gamma_0$ ,

$$\gamma_0(r) = \tilde{\gamma}_0(r) + L\,\mathrm{sign}^+(r) \quad , \quad r \in \mathbb{R} \, , \tag{1.3}$$

where $\tilde{\gamma}_0(r) = \int_0^r \rho(\xi)\,d\xi$ , $\rho(r) = \frac{c(r)}{k(r)} \geqq 0$ , $L \geqq 0$ ,

$$\mathrm{sign}^+(r) = \begin{cases} 0 & \text{if} \quad r < 0 \\ [0,1] & \text{if} \quad r = 0 \\ 1 & \text{if} \quad r > 0 \, . \end{cases}$$

In heat conduction processes with phase transition, $\vartheta$ represents temperature, $\lambda$ distributed heat sources, L latent heat of phase transition, and c,k temperature-dependent specific heat and heat conductivity, respectively. Both c and k are assumed smooth up to finite jumps at $\vartheta = 0$ (critical point of phase transition); k is strictly positive throughout while c nonnegative, possibly vanishing at some values of $\vartheta$ . Hence, equation (1.2a) is of the mixed parabolic-elliptic type. In (1.2b), $p = p(x)$, $x \in \Gamma$ represents the heat permeability of $\Gamma$ and $u = u(x,t)$ is the boundary control. Problem (1.2),(1.3) describes also saturated-unsaturated flows in porous media and electrochemical machining processes (cf., [5,7]).

It is to be noted that in the parabolic case ($\rho > 0$) graph $\gamma_0$ is strictly monotone, while in the parabolic-elliptic case ($\rho \geqq 0$) it is only monotone (not strictly).

For heat conduction processes, the control objective (1.1) is to approach a given temperature distribution $\vartheta_d \in L^2(Q)$ by means of boundary controls $u \in U$ of possibly small norms; $\alpha$ is an arbitrary weight coefficient which is related to the energy cost of the control. We remark that the control space $U^0$ is representative for numerous applications, e.g., control by means of $L^2$-boundary flux or temperature of environment. Also $u \in U^1$ is a natural way of acting for many real problems, for instance with the environment temperature specified by heat power of external sources (control $u$ is then the solution of an additional ordinary differential equation).

Now we specify the notion of solution to Stefan problem (1.2),(1.3) by introducing a relevant variational inequality formulation in terms of the freezing index $y(x,t) = \int_0^t \vartheta(x,s)\,ds$,

$$\begin{cases} \gamma_0(y') - \Delta y \ni f_0 & \text{in } Q \,, & (1.4a) \\ \partial_\nu y + p\,y = g & \text{on } \Sigma \,, & (1.4b) \\ y(0) = 0 & \text{in } \Omega \,, & (1.4c) \end{cases}$$

with $f_0(x,t) = \tilde{f}(x,t) + w_0(x)$ , $\tilde{f}(x,t) = \int_0^t \lambda(x,s)ds$, $g(x,t) = \int_0^t u(x,s)ds$.

System (1.4) admits the following variational inequality formulation (cf., [9] ) :

<u>(VI).</u> Determine $y \in H^1(0,T;V)$ such that

$$\begin{cases} F_1(y(t),z;\tilde{\gamma}_0,\Psi_0,f_0(t),g(t)) \equiv (\tilde{\gamma}_0(y'(t))-f_0(t),z-y'(t)) + \\ \quad + a(y(t),z-y'(t)) - (g(t),z-y'(t))_\Gamma + \Psi_0(z) - \\ \quad - \Psi_0(y'(t)) \geqq 0 \,, \quad \forall\, z \in V \,, \quad \text{a.a. } t \in [0,T] \,, & (1.5a) \\ y(0) = 0 \quad \text{in } \Omega \,, & (1.5b) \end{cases}$$

with : $V = H^1(\Omega)$, $H = L^2(\Omega)$, $\|\cdot\|_V$ , $\|\cdot\|_H$ the corresponding standard norms; $(\cdot,\cdot)$, $(\cdot,\cdot)_\Gamma$ standard scalar products in $H$ and $L^2(\Gamma)$, respectively;

$$a(y,z) = (\nabla y, \nabla z) + (py,z)_\Gamma \,,$$

$$\Psi_0(z) = L \int_\Omega \psi_0(z(x))\,dx \,, \qquad \psi_0(z) = \max\{0,z\} \,.$$

DEFINITION. By the <u>weak solution</u> of the Stefan problem (1.2), (1.3) we mean a function $y$ (which represents the freezing index of system) such that variational inequality (VI) is satisfied.

Since $y' = \vartheta$ a.e. in $Q$ , $y'$ (or $y$ ) can be treated as the state variable of the system, corresponding to control $u$. The control problem under study admits thus the formulation

<u>(CP).</u> Minimize $J(y',u)$ over $y' \in L^2(Q)$ and $u \in \mathcal{U}$ , subject to $y = y(u)$ which satisfies (VI) with $u$ .

## 2. BASIC STRUCTURAL PROPERTIES OF VARIATIONAL INEQUALITY (VI)

### 2.1. Underlying hypotheses

(A1) $\rho$ admits representation $\rho(r) = \tilde{\rho}(r) + \tilde{\tilde{\rho}}\,\mathrm{sign}^+(r)$ , where $\tilde{\rho} \in C^1(\mathbb{R})$ and $\tilde{\tilde{\rho}}$ finite constant; $0 \leqq \bar{\rho} \leqq \rho(r) \leqq \bar{\bar{\rho}} < +\infty$, $r \in \mathbb{R}$.

Two cases are to be distinguished: (i) parabolic if $\bar{\rho} > 0$ , (ii) degenerate (parabolic-elliptic) if $\bar{\rho} = 0$ .

$(A2)^0$ $\lambda \in L^2(Q)$ ; $(A2)^1$ $\lambda \in H^1(0,T;H)$ ;

(A3) $\vartheta_0 \in V \cap L^\infty(\Omega)$ , $w_0 \in H$ , $w_0 = (\gamma_0)^0(\vartheta_0)$ , where $(\gamma_0)^0$ is the minimum-norm section of the graph $\gamma_0$ ; $L \geqq 0$ ;

(A4) $p \in L^\infty(\Gamma)$ , $p \geqq 0$ a.e. on $\Gamma$ , the set $\{ x \in \Gamma \mid p(x) > 0 \}$ has positive Lebesgue measure in $\Gamma$ ;

$(A5)^0$ $u \in \mathcal{U}^0$ ; $(A5)^1$ $u \in \mathcal{U}^1$ .

By (A1) , the mapping $\tilde{\gamma}_0 : H \to H$ is Lipschitz continuous with Lipschitz constant $\bar{\bar{\rho}}$ , and monotone (strictly if $\bar{\rho} > 0$),

$$(\tilde{\gamma}_0(y) - \tilde{\gamma}_0(z), y - z) \geqq \bar{\rho}\, \|y - z\|_H^2 , \quad \forall\, y,z \in H . \tag{2.1}$$

Functional $\Psi_0 : V \to \mathbb{R}$ is bounded, convex, l.s.c.; bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is symmetric, continuous and V-elliptic.

### 2.2. Regularization

We introduce auxiliary regularized problems which approximate (VI). The procedure comprises parabolic regularization of the problem (if $\bar{\rho} = 0$ ) and its smoothing. The first is based on approximating $\gamma_0$ by strictly monotone graphs $\gamma_\mu$ , $\mu \in (0,1]$, in the other $\gamma_\mu$ is approached by smooth functions $\gamma_{\mu\varepsilon}$, $\varepsilon \in (0,1]$.

#### 2.2.1. Parabolic regularization

In order to tackle simultaneously the parabolic and degenerate cases, we modify (1.4) by replacing $\gamma_0$ with

$$\gamma_\mu(r) = \tilde{\gamma}_\mu(r) + L\,\mathrm{sign}^+(r)\ , \quad r \in \mathbb{R}\ , \quad \mu \in [0,1]\ , \tag{2.2}$$

where $\tilde{\gamma}_\mu(r) = \int_0^r \rho_\mu(\xi)\,d\xi$ , $\rho_\mu(r) = \rho(r) + \mu$ .

Clearly, at $\mu = 0$ , $\gamma_\mu = \gamma_o$ . We also have

$$0 \leqq \bar{\rho} + \mu = \bar{\rho}_\mu \leqq \rho_\mu(r) \leqq \bar{\bar{\rho}}_\mu = \bar{\bar{\rho}} + \mu < +\infty\ , \quad \forall\ r \in \mathbb{R}\ .$$

Correspondingly, we replace $f_o$ in (1.4) by

$$f_\mu(x,t) = \tilde{f}(x,t) + w_\mu(x)\ , \quad w_\mu(x) = w_o(x) + \mu\,\vartheta_o(x), \tag{2.3}$$

to get the variational inequality

<u>$(VI)^\mu$, $\mu \in [0,1]$.</u> Determine $y_\mu \in H^1(0,T;V)$ , such that

$$\begin{cases} F_1(y_\mu(t),z;\tilde{\gamma}_\mu,\Psi_o,f_\mu(t),g(t)) \geqq 0\ , \quad \forall\ z \in V\ , \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.a. } t \in [0,T], & (2.4a) \\ y_\mu(0) = 0 \quad \text{in } \Omega\ . & (2.4b) \end{cases}$$

Clearly, $(VI)^\mu$ with $\mu = 0$ coincides with (VI). Notice also that $\tilde{\gamma}_\mu : H \to H$ is Lipschitz continuous with the constant $\bar{\bar{\rho}}_\mu$ , and satisfies (2.1) with $\bar{\rho}_\mu$ . If $\bar{\rho}_\mu > 0$ , then $(VI)^\mu$ is parabolic (or parabolically regularized); if $\bar{\rho}_\mu = 0$ , the problem is degenerate.

Further, it will be useful to notice that problem $(VI)^\mu$ admits some alternative formulations.

LEMMA 2.1. (2.4a) is equivalent to the inequality

$$\begin{aligned} F_2(y_\mu(t),z;F^\mu,f_\mu(t),g(t)) &\equiv a(y_\mu(t),z - y_\mu'(t)) - \\ &- (f_\mu(t),z - y_\mu'(t)) - (g(t),z - y_\mu'(t))_\Gamma + F^\mu(z) - \\ &- F^\mu(y_\mu'(t)) \geqq 0\ , \quad \forall\ z \in V\ , \quad \text{a.a. } t \in [0,T]\ , \end{aligned} \tag{2.5}$$

where $F^\mu(z) = B^\mu(z) + \Psi_o(z)$ , $B^\mu(z) = \int_\Omega \beta^\mu(z(x))\,dx$ ,

$$\beta^\mu(z) = \int_0^z \tilde{\gamma}_\mu(\xi)\,d\xi\ .$$

<u>Proof.</u> The Gateaux differential $DB^\mu$ of functional $B^\mu : H \to \mathbb{R}$ admits the characterization

$$(DB^{\mu}(y),z) = (\tilde{\gamma}_{\mu}(y),z) \ , \quad \forall\ y,z \in H \ , \tag{2.6}$$

hence, due to the monotonicity (strict if $\bar{\rho}_{\mu} > 0$ ) of $\tilde{\gamma}_{\mu}$ , $B^{\mu}$ is convex (strictly convex). Consequently,

$$(\tilde{\gamma}_{\mu}(y), z - y) \leqq B^{\mu}(z) - B^{\mu}(y) \ , \quad \forall\ y,z \in H \ ,$$

thus (2.4a) implies (2.5). To show the converse, take $z = y_{\mu}'(t) + \delta(\bar{z} - y_{\mu}'(t))$, $\delta \in (0,1)$, $\bar{z} \in V$, in (2.5). Then, because of the convexity of $\Psi_o$ ,

$$a(y_{\mu}(t),\bar{z} - y_{\mu}'(t)) - (f_{\mu}(t),\bar{z} - y_{\mu}'(t)) - (g(t),\bar{z} - y_{\mu}'(t))_{\Gamma} +$$

$$+ \frac{1}{\delta} \left[B^{\mu}(y_{\mu}'(t) + \delta(\bar{z} - y_{\mu}'(t))) - B^{\mu}(y_{\mu}'(t))\right] + \Psi_o(z) -$$

$$- \Psi_o(y_{\mu}'(t)) \geqq 0 \ , \quad \forall\ \bar{z} \in V \ .$$

Upon letting $\delta \to 0$ in the above inequality, due to (2.6) we get (2.4a). □

<u>Remark 2.1.</u> For any $\mu \in [0,1]$, functional $F^{\mu} : V \to \mathbb{R}$ is convex (strictly convex if $\bar{\rho}_{\mu} > 0$ ), weakly l.s.c. on $V$ , and lower bounded, $F^{\mu}(z) \geqq \frac{\bar{\rho}_{\mu}}{2} \|z\|_H^2$ , $\forall\ z \in H$ . Moreover,

$$\lim_{\mu \to 0} \int_0^T F^{\mu}(z(t))\,dt = \int_0^T F(z(t))\,dt \ , \quad \forall\ z \in L^2(0,T;V) \ ; \tag{2.7a}$$

if $z_{\mu} \to z$ weakly in $L^2(0,T;V)$, then

$$\liminf_{\mu \to 0} \int_0^T F^{\mu}(z_{\mu}(t))\,dt \geqq \int_0^T F(z(t))\,dt \ . \tag{2.7b}$$

<u>Remark 2.2.</u> $(VI)^{\mu}$ is equivalent to the time-integrated variational inequality

$$\int_0^T \mathcal{F}_2(y_{\mu}(t),z(t);F^{\mu},f_{\mu}(t),g(t))\,dt \geqq 0 \ , \quad \forall\ z \in L^2(0,T;V), \tag{2.8}$$

with initial condition (2.4b).

### 2.2.2. <u>Smooth approximation</u>

To approximate graph $\gamma_{\mu}$ , we define the following single-valued functions

$$\gamma_{\mu\varepsilon}(r) = \tilde{\gamma}_{\mu\varepsilon}(r) + L\,\chi_\varepsilon(r)\ , \qquad r \in \mathbb{R}\ , \qquad \varepsilon \in (0,1]\ , \qquad (2.9)$$

with $\tilde{\gamma}_{\mu\varepsilon}(r) = \int_0^r \rho_{\mu\varepsilon}(\xi)\,d\xi$ , $\rho_{\mu\varepsilon}(r) = \tilde{\rho}(r) + \tilde{\tilde{\rho}}\,\chi_\varepsilon(r) + \mu$ .

$\chi_\varepsilon(\cdot)$ is a $C^2$-approximation of $\operatorname{sign}^+(\cdot)$ (for instance in a polynomial form, cf., [9]). Then $\gamma_{\mu\varepsilon} \in C^2(\mathbb{R})$ and $\gamma_{\mu\varepsilon}$ approximates graph $\gamma_\mu$ in sense of the uniform convergence on compact subsets of $\mathbb{R}\setminus\{0\}$ . Moreover,

$$D\gamma_{\mu\varepsilon}(r) = D\gamma_\mu(r)\ , \qquad r \in (-\infty,0\,]\cup[\,\varepsilon,+\infty)\ ,$$

$$\bar{\rho}_\mu \leqq D\gamma_{\mu\varepsilon}(r) \leqq \frac{C}{\varepsilon}\ , \qquad \tilde{\rho}_\mu \leqq D\tilde{\gamma}_{\mu\varepsilon}(r) \leqq \bar{\bar{\rho}}_\mu\ ,$$

$$|\,D^2\gamma_{\mu\varepsilon}(r)\,| \leqq \frac{C}{\varepsilon^2}\ , \qquad \forall\ r \in \mathbb{R}\ , \qquad (2.10)$$

where $C$ is a constant independent of $\mu,\varepsilon$ . By (A3), the introduced approximations induce the following compatible smooth approximations to $f_\mu$ ,

$$f_{\mu\varepsilon}(x,t) = \tilde{f}(x,t) + w_{\mu\varepsilon}(x)\ , \quad w_{\mu\varepsilon}(x) = \gamma_{\mu\varepsilon}(\vartheta_0(x)). \quad (2.11a)$$

Notice that $\|w_{\mu\varepsilon}\|_{L^\infty(\Omega)} \leqq C$ with a constant $C$ independent of $\mu,\varepsilon$ ; hence, for any $\mu \geqq 0$ ,

$$w_{\mu\varepsilon} \to w_\mu \qquad \text{a.e. in } \Omega \text{ as } \varepsilon \to 0\ . \qquad (2.11b)$$

The corresponding smooth approximation of problem (1.4) takes the form

$$\begin{cases} \gamma_{\mu\varepsilon}(y'_{\mu\varepsilon}) - \Delta y_{\mu\varepsilon} = f_{\mu\varepsilon} \quad \text{in } Q\ , \\ (1.4b)\ ,\ (1.4c)\ , \end{cases} \qquad (2.12)$$

which gives rise to the following variational inequality

<u>$(VI)^\mu_\varepsilon$ , $\mu \in [0,1]$ , $\varepsilon \in (0,1]$.</u> Determine $y_{\mu\varepsilon} \in H^1(0,T;V)$ such that

$$\begin{cases} \mathcal{F}_1(y_{\mu\varepsilon}(t),z;\tilde{\gamma}_{\mu\varepsilon},\Psi_\varepsilon,f_{\mu\varepsilon}(t),g(t)) \geqq 0\ , \quad \forall\ z \in V\ , \\ \qquad\qquad \text{a.a. } t \in [0,T]\ , \quad (2.13a) \\ y_{\mu\varepsilon}(0) = 0 \quad \text{in } \Omega\ , \quad (2.13b) \end{cases}$$

where $\Psi_\varepsilon(z) = L \int_\Omega \psi_\varepsilon(z(x))\,dx$ , $\psi_\varepsilon(r) = \int_0^r \chi_\varepsilon(\xi)\,d\xi$ , i.e., $\psi_\varepsilon \in C^3(\mathbb{R})$ and uniformly approximates $\psi_o$. For any $\varepsilon > 0$, functionals $\Psi_\varepsilon : V \to \mathbb{R}$ are bounded, convex, l.s.c. and Gateaux differentiable; besides, convergences analogous to (2.7) hold as $\varepsilon \to 0$ .

Remark 2.3. Variational inequality (2.13a) admits equivalent formulation in the form (2.5), with $F^\mu$, $f_\mu$ to be respectively replaced by $F^\mu_\varepsilon$ and $f_{\mu\varepsilon}$ , where

$$F^\mu_\varepsilon(z) = B^\mu_\varepsilon(z) + \Psi_\varepsilon(z) ,$$

$$B^\mu_\varepsilon(z) = \int_\Omega \beta^\mu_\varepsilon(z(x))\,dx , \qquad \beta^\mu_\varepsilon(r) = \int_0^r \tilde{\gamma}_{\mu\varepsilon}(\xi)\,d\xi .$$

Clearly, $F^\mu_\varepsilon : V \to \mathbb{R}$ preserves properties of $F^\mu$. Besides, the convergences analogous to (2.7) hold, equally at $\varepsilon \to 0$ (for any fixed $\mu \in [0,1]$ ) and at $\mu, \varepsilon \to 0$ .

Remark 2.4. Further on, $(VI)^\mu_\varepsilon$ with $\varepsilon = 0$ will be identified with $(VI)^\mu$. For simplicity, we shall omit indices $\mu, \varepsilon$ in all subsequent notations, whenever equal zero.

## 2.3. Existence and uniqueness

Consider variational inequality (VI). Because of our primary interest in control problem (CP) stated for (VI), we shall discuss properties of the solutions to (VI) for various classes of admissible controls, with possible degeneracy of the problem itself.

THEOREM 2.1 ( $\bar{\rho}_\mu > 0$ , $u \in \mathcal{U}^1$ ). Let (A1),$(A2)^o$,(A3),(A4) be satisfied. Then there exists a unique solution $y_\mu \in W^{1,\infty}(0,T;V) \cap H^2(0,T;H)$ of $(VI)^\mu$, $\mu \in [0,1]$, such that $y'_\mu(0) = \vartheta_o$ in $\Omega$ . Moreover, the following a priori bounds hold:

$$\|y_\mu\|_{H^1(0,T;V)} + \bar{\rho}_\mu^{1/2} \|y'_\mu\|_{L^\infty(0,T;H)} \leqq C_o , \tag{2.14a}$$

$$\|y'_\mu\|_{L^\infty(0,T;V)} + \bar{\rho}_\mu^{1/2} \|y''_\mu\|_{L^2(Q)} \leqq C_1 \quad \text{if } \bar{\rho} > 0, \tag{2.14b}$$

and, provided $(A2)^1$,

$$\|y'_\mu\|_{L^\infty(0,T;V)} + \mu^{1/2}\|y''_\mu\|_{L^2(Q)} \leqq C_2 \;, \quad \text{if} \quad \bar\rho = 0 \;, \quad (2.14c)$$

with positive constants $C_o, C_1, C_2$ dependent on the following data:

$$C_o = C_o(\|\lambda\|_{L^2(Q)}, \|\vartheta_o\|_H, \|u\|_{U^o}) \;,$$

$$C_1 = C_1(\|\lambda\|_{L^2(Q)}, \|\vartheta_o\|_V, \|u\|_{U^1}) \;,$$

$$C_2 = C_2(\|\lambda\|_{H^1(0,T;H)}, \|\vartheta_o\|_V, \|u\|_{U^1}) \;. \qquad (2.15)$$

<u>Proof</u> (outline; cf., [9] for details). A Galerkin approximation to the regularized variational inequality $(VI)^\mu_\varepsilon$, $\varepsilon \in (0,1]$, is constructed. Let $\{v_1,\dots,v_m\}$ be a system of linearly independent elements in $V$, such that $cl(\bigcup_{m\in\mathbb{N}} V_m) = V$, for $V_m = \mathrm{span}\{v_1,\dots,v_m\}$. The elements $v_1, v_2$ are to be selected so that $0, \vartheta_o \in \mathrm{span}\{v_1,v_2\}$. The following family of approximating semidiscrete Galerkin problems is introduced:

$$\begin{cases} \text{Determine} \quad y_m = y_{\mu\varepsilon m} \in H^1(0,T;V_m) \quad (m \geqq 2), \quad \text{such that} \\ (\gamma_{\mu\varepsilon}(y'_m(t)) - f_{\mu\varepsilon}(t), z_m) + a(y_m(t), z_m) - (g(t), z_m)_\Gamma = 0 \;, \\ \qquad\qquad \forall\, z_m \in V_m \;, \quad \text{a.a. } t \in [0,T], \quad (2.16) \\ y_m(0) = 0 \quad \text{in } \Omega \;. \end{cases}$$

System (2.16) of nonlinear ordinary differential equations admits a solution $y_m$ over $[0,T]$, which satisfies bounds (2.14) uniformly in $m, \varepsilon$. This enables us to pass in (2.16) to the limit with $m \to \infty$ ($\varepsilon$ fixed), and to show that the resulting limit function $y_{\mu\varepsilon}$ satisfies $(VI)^\mu_\varepsilon$. Next, due to the analogous uniform bounds for $y_{\mu\varepsilon}$, we pass in $(VI)^\mu_\varepsilon$ to the limit with $\varepsilon \to 0$, and conclude that the relevant limit $y_\mu$ is a solution to $(VI)^\mu$. Estimates (2.14) on $y_{\mu\varepsilon}$ imply the same bounds for $y_\mu$.

The uniqueness follows directly from the stability of solutions to $(VI)^\mu$ with respect to perturbations of the data as shown in [9]. □

THEOREM 2.2 ($\bar\rho_\mu > 0$, $u \in U^o$). Let (A1), $(A2)^o$, (A3), (A4) be satisfied. Then there exists a unique solution $y_\mu \in H^1(0,T;V) \cap W^{1,\infty}(0,T;H)$ of $(VI)^\mu$, $\mu \in [0,1]$, which satisfies the bound

(2.14a). This solution may be constructed as a limit of the solutions $y_{\mu n}$ to problems $(VI)^{\mu,n}$ which correspond to $u_n \in \mathcal{U}^1$ :

$$\text{if} \quad u_n \to u \quad \text{strongly in } \mathcal{U}^o \text{ as } n \to \infty \text{, then} \tag{2.17a}$$

$$\begin{aligned} y_{\mu n} \to y_\mu \quad & \text{weakly in } H^1(0,T;V) \text{ ,} \\ & \text{weakly-* in } W^{1,\infty}(0,T;H) \text{ ,} \end{aligned} \tag{2.17b}$$

$$y'_{\mu n} \to y'_\mu \quad \text{strongly in } L^2(Q) \text{ ,} \tag{2.17c}$$

where $y_\mu$ is the solution of $(VI)^\mu$ corresponding to $u \in \mathcal{U}^o$.

<u>Proof.</u> Since $\mathcal{U}^1$ is dense in $\mathcal{U}^o$, for any $u \in \mathcal{U}^o$ there exists a sequence $\{u_n\} \subset \mathcal{U}^1$ which satisfies (2.17a). Let $\{y_{\mu n}\}$ be the sequence of solutions to $(VI)^{\mu,n}$ with $u_n$ . Due to (2.14a) and the boundedness of $\{u_n\}$ in $\mathcal{U}^o$ , $\{y_{\mu n}\}$ is uniformly bounded in $H^1(0,T;V) \cap W^{1,\infty}(0,T;H)$ . For a subsequence, this yields (2.17b).

To get (2.17c), it is enough to show that $\{y'_{\mu n}\}$ is a Cauchy sequence in $L^2(Q)$ . To this end, consider problems $(VI)^{\mu,n}$ and $(VI)^{\mu,m}$, respectively corresponding to $u_n$ and $u_m$ . Take $z = y'_{\mu n}(t)$ in the inequality (2.4a) with $u_m$ , and $z = y'_{\mu m}(t)$ in the same inequality with $u_n$ . By combining both inequalities integrated over $[0,t]$ , $0 < t \leqq T$ , due to the strict monotonicity of $\tilde{\gamma}_\mu$ we get

$$\begin{aligned} \bar{\rho}_\mu \int_0^t \| y'_{\mu n}(\tau) - y'_{\mu m}(\tau) \|_H^2 d\tau \, + \\ + \frac{1}{2} a(y_{\mu n}(t) - y_{\mu m}(t), y_{\mu n}(t) - y_{\mu m}(t)) \leqq \\ \leqq \int_0^t \left( \int_0^\tau (u_n(s) - u_m(s)) ds \, , y'_{\mu n}(\tau) - y'_{\mu m}(\tau) \right)_\Gamma d\tau \, . \end{aligned} \tag{2.18}$$

Hence, by the uniform boundedness of $y'_{\mu n}|_\Sigma$ in $L^2(\Sigma)$, we see that indeed $\{y'_{\mu n}\}$ is a Cauchy sequence in $L^2(Q)$. To complete the proof, it remains to show that the limit $y_\mu$ is solution of $(VI)^\mu$ with $u$ . To this purpose, take the upper limit at $n \to \infty$ in the inequality (2.8) with $u_n$ . By (2.17) and the weak l.s.c. of $F^\mu$ , it follows that $y_\mu$ satisfies $(VI)^\mu$ corresponding to $u$ . Furthermore, the bounds (2.14a) on $y_{\mu n}$ imply their analogues for $y_\mu$ . With the uniqueness following as in Thm. 2.1, this completes the proof. $\square$

By applying the parabolic regularization $(VI)^\mu$, $\mu > 0$, a relevant result can be proved for (VI) in the degenerate case, equally at u in $\mathcal{U}^1$ and $\mathcal{U}^o$.

THEOREM 2.3 ( $\bar{\rho} = 0$ ).
(I) Let (A1), $(A2)^1$, (A3), (A4), $(A5)^1$ hold. Then there exists a unique solution $y \in W^{1,\infty}(0,T;V)$ to (VI); this solution satisfies the bounds

$$\|y\|_{H^1(0,T;V)} \leqq C_o \quad , \quad \|y\|_{W^{1,\infty}(0,T;V)} \leqq C_2 \tag{2.19}$$

with the same constants $C_o, C_2$ as in (2.15). It may be constructed by taking the limit of solutions $y_\mu$ to $(VI)^\mu$ at $\mu \to 0$,

$$y_\mu \to y \quad \text{weakly-* in } W^{1,\infty}(0,T;V) \, , \tag{2.20a}$$

$$y'_\mu \to y' \quad \text{strongly in } L^2(Q) \, ; \tag{2.20b}$$

moreover,

$$\mu^{1/2} y'_\mu \to 0 \quad \text{strongly in } L^\infty(0,T;V) \, ,$$

$$\mu y''_\mu \to 0 \quad \text{strongly in } L^2(Q) \, . \tag{2.20c}$$

(II) Let (A1), $(A2)^o$, (A3), (A4) and $(A5)^o$ hold. Then there exists a unique solution $y \in H^1(0,T;V)$ of (VI); it satisfies the bound

$$\|y\|_{H^1(0,T;V)} \leqq C_o \tag{2.21}$$

with the same constant $C_o$ as in (2.15). This solution may be constructed by taking the limit of solutions $y_\mu$ to $(VI)^\mu$ at $\mu \to 0$,

$$y_\mu \to y \quad \text{weakly in } H^1(0,T;V) \, , \tag{2.22a}$$

$$y'_\mu \to y' \quad \text{strongly in } L^2(Q) \, ; \tag{2.22b}$$

moreover,
$$\mu y'_\mu \to 0 \quad \text{strongly in } L^\infty(0,T;H) \, . \tag{2.22c}$$

<u>Proof.</u> (I) By estimates (2.14a,c) for $\{y_\mu\}$, the convergences (2.20a,c) follow directly. To show (2.20b), consider problems $(VI)^\mu$ and $(VI)^\lambda$ with $\mu, \lambda \in (0,1]$. Take $z = y'_\lambda(t)$ in (2.4a) and $z = y'_\mu(t)$ in the same inequality corresponding to $\lambda$. By combining both inequalities integrated over $[0,t]$, $0 < t \leqq T$,

due to the monotonicity of $\tilde{\gamma}_o$ , we get

$$\int_0^t ( \mu(y'_\mu(s)-\vartheta_o) - \lambda(y'_\lambda(s)-\vartheta_o) , y'_\mu(s)-y'_\lambda(s) ) \, ds + $$
$$+ \frac{1}{2} a(y_\mu(t)-y_\lambda(t), y_\mu(t)-y_\lambda(t)) \leqq 0 , \qquad t \in (0,T] .$$

Hence, in particular

$$( \mu(y'_\mu-\vartheta_o) - \lambda(y'_\lambda-\vartheta_o) , y'_\mu - y'_\lambda )_{L^2(Q)} \leqq 0 , \quad \forall \ \mu,\lambda \in (0,1]. \tag{2.23}$$

Due to the Crandall-Pazy lemma [4] , the boundedness of $\{y'_\mu\}$ in $L^2(Q)$ together with (2.23) imply that $\|y'_\mu\|_{L^2(Q)}$ is nondecreasing in $\mu$ , and

$$y'_\mu \to y' \quad \text{strongly in } L^2(Q) \quad \text{as} \quad \mu \to 0 .$$

It remains to show that $y$ satisfies (VI). To this end, take the upper limit at $\mu \to 0$ in the inequality (2.4a) integrated in time. Due to (2.20), continuity of $\tilde{\gamma}_o$ and weak l.s.c. of $\Psi_o$ , it follows that $y$ is a solution to (VI). Estimates (2.19) are direct consequences of their counterparts for $y_\mu$ . The uniqueness follows as in [9] .

The proof of (II) proceeds much the same way as for (I). □

## 3. STRUCTURAL PROPERTIES OF CONTROL PROBLEM (CP)

### 3.1. <u>State observation mapping</u>

Let $\Xi : \mathcal{U} \to L^2(Q)$ denote the state observation mapping defined by $\Xi(u) = y'$ , where $y$ is the solution of (VI) with control $u$ . Then, control problem (CP) admits the equivalent formulation in the control space,

<u>(CP).</u> $\quad \inf\limits_{u \in \mathcal{U}} \{ I(u) = J(\Xi(u),u) \}$ .

The observation mapping $\Xi$ is continuous in the following sense.

PROPOSITION 3.1. Assume that (A1),(A2)$^o$ ,(A3),(A4) and, alternatively, $(A5)^{o}$ or $(A5)^{1}$ holds. Then

(I) degenerate case ($\bar{\rho} = 0$) :

(i) $\Xi$ is continuous from $\mathcal{U}$(weak) into $L^2(0,T;V)$(weak);

(II) parabolic case ($\bar{\rho} > 0$) : (i) holds, furthermore

(ii) $\Xi$ is compact from $\mathcal{U}^{o}$ into $L^2(Q)$;

(iii) $\Xi$ is Lipschitz continuous from $\mathcal{U}^{o}$ into $L^2(Q)$.

Proof. (I) Consider any sequence $\{u_n\} \subset \mathcal{U}$, such that $u_n \to u$ weakly in $\mathcal{U}$. Let $\{y_n\}$ be the sequence of solutions to (VI) which correspond to $\{u_n\}$. Since $\{u_n\}$ is bounded in $\mathcal{U}$, it follows by (2.21) that $\{y_n\}$ is uniformly bounded in $H^1(0,T;V)$. Therefore, for a subsequence,

$$y_n \to y \quad \text{weakly in } H^1(0,T;V) . \tag{3.1}$$

To show that $y$ is the solution of (VI) corresponding to $u$, we pass to the limit as $n \to \infty$ in (VI) with $u_n$. Notice that, according to (3.1), $y(0) = 0$ in $\Omega$ ; besides, as $n \to \infty$,

$$\int_0^T \Big( \int_0^t u_n(s)\,ds\,,\, y_n'(t) \Big)_\Gamma dt = - \int_0^T (u_n(t), y_n(t))_\Gamma\, dt + \\ + \Big( \int_0^T u_n(s)\,ds\,,\, y_n(T) \Big)_\Gamma \to \int_0^T \Big( \int_0^t u(s)\,ds\,,\, y'(t) \Big)_\Gamma dt. \tag{3.2}$$

Take the upper limit as $n \to \infty$ in (2.8) at $\mu = 0$, with $u_n$. By (3.2) and due to the weak l.s.c. of $F$, we can conclude that $y$ satisfies (2.8) with $u_n$ replaced by $u$. Thus, $y$ is the solution of (VI), corresponding to $u$. Hence, (i) has been proved.

(II) In the parabolic case, assertion (ii) follows as in the proof of Thm. 2.2 (cf., (2.18) ). Indeed, from integrating by parts the right-hand side of (2.18) we get

$$\bar{\rho}\, \|y_n' - y_m'\|^2_{L^2(Q)} + \frac{1}{2} a(y_n(T)-y_m(T), y_n(T)-y_m(T)) \leqq \\ \leqq - \int_0^T (u_n(t)-u_m(t), y_n(t)-y_m(t))_\Gamma\, dt + \\ + \Big( \int_0^T (u_n(s)-u_m(s))\,ds\,,\, y_n(T)-y_m(T) \Big)_\Gamma\,, \quad \forall\, n,m \in \mathbb{N}\,. \tag{3.3}$$

Hence, due to (3.1), $\{y_n'\}$ is a Cauchy sequence in $L^2(Q)$. Thus,

$y'_n \to y'$ strongly in $L^2(Q)$.

To conclude (iii), we use inequality (2.18) with the right-hand side integrated by parts. An appropriate use of Young's and Gronwall's inequalities yields then the estimate

$$\|y'_n - y'_m\|_{L^2(Q)} + \|y_n - y_m\|_{L^\infty(0,T;V)} \leqq C\,\|u_n - u_m\|_{L^2(\Sigma)}, \tag{3.4}$$

with a positive constant $C$ independent of $u_n$, $u_m$ (dependent on $\bar{\rho}^{-1}$). The proof is complete. □

## 3.2. Existence of optimal solutions

A direct consequence of the continuity of $\Xi$ is

THEOREM 3.1. Control problem (CP) has nonempty set of optimal solutions.

Proof. Let $\{u_u\} \subset \mathcal{U}$ be a minimizing sequence for functional $I$, i.e.,

$$\lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} J(\Xi(u_n),u_n) = \hat{I} \equiv \inf_{u\in\mathcal{U}} I(u) .$$

Hence, by the radial unboundedness of functional $I$, the sequence $\{u_n\}$ is uniformly bounded in $\mathcal{U}$. Therefore, for a subsequence, $u_n \to \hat{u}$ weakly in $\mathcal{U}$ as $n\to\infty$. Due to Prop. 3.1,(i) we get immediately

$$J(\Xi(\hat{u}),\hat{u}) \leqq \liminf_{n\to\infty} J(\Xi(u_n),u_n) = \hat{I} ,$$

and hence conclude that $\hat{u}$ is an optimal control for (CP). □

## 3.3. Role of control spaces $\mathcal{U}^o$ and $\mathcal{U}^1$

There is a link between problems with control spaces $\mathcal{U}^o$ and $\mathcal{U}^1$, which reflects a regularizing role of controls from the space $\mathcal{U}^1$ against those from $\mathcal{U}^o$.

THEOREM 3.2. Consider the family of control problems

$(CP)_\nu$, $\nu \geqq 0$. $\quad \inf \{ I_\nu(u) = \|\Xi(u)-\vartheta_d\|^2_{L^2(Q)} + \|u\|^2_{\mathcal{U}^o} +$

$$+ \nu \|u'\|^2_{\mathcal{U}^o} \} ,$$

over $u \in \mathcal{U}^1$ if $\nu > 0$, and over $u \in \mathcal{U}^o$ if $\nu = 0$.

Assume that (A1),(A2)$^o$,(A3),(A4),(A5)$^o$ hold and $\bar{\rho} > 0$. Then there exists a sequence $\{\hat{u}_\nu\} \subset \mathcal{U}^1$ of optimal controls to $(CP)_\nu$, such that as $\nu \to 0$,

$$\hat{u}_\nu \to \hat{u} \quad \text{strongly in } \mathcal{U}^o , \tag{3.5}$$

$$\Xi(\hat{u}_\nu) \to \Xi(\hat{u}) \quad \text{weakly in } L^2(0,T;V) , \quad \text{strongly in } L^2(Q) , \tag{3.6}$$

$$\hat{I}_\nu \to \hat{I}_o , \tag{3.7}$$

where $\hat{u} \in \mathcal{U}^o$ is an optimal control to $(CP)_o$ ; $\hat{I}_\nu = I_\nu(\hat{u}_\nu)$, $\hat{I}_o = I_o(\hat{u})$.

<u>Proof.</u> Notice that, for each $\nu > 0$,

$$\hat{I}_\nu = \inf_{u \in \mathcal{U}^1} I_\nu(u) \geqq \inf_{u \in \mathcal{U}^o} I_o(u) = \hat{I}_o . \tag{3.8}$$

Now, we show that for every $\sigma > 0$ there is $\nu(\sigma)$ such that

$$\hat{I}_\nu \leqq \hat{I}_o + \sigma , \quad \text{for } \nu \leqq \nu(\sigma) . \tag{3.9}$$

Observe that, for any pair $u_1, u_2 \in \mathcal{U}^o$, due to the Lipschitz continuity and boundedness of $\Xi$ (cf., (3.4),(2.14a) ),

$$|I_o(u_1) - I_o(u_2)| \leqq C \|u_1 - u_2\|_{\mathcal{U}^o} , \quad C \equiv C( \|u_1\|_{\mathcal{U}^o}, \|u_2\|_{\mathcal{U}^o}). \tag{3.10}$$

By the density of $\mathcal{U}^1$ in $\mathcal{U}^o$, (3.10) implies the existence of $w \in \mathcal{U}^1$ such that

$$|I_o(w) - I_o(\hat{u}_o)| \leqq C \|w - \hat{u}_o\|_{\mathcal{U}^o} \leqq \frac{\sigma}{2} ,$$

where $\hat{u}_o$ is any optimal control for $(CP)_o$; consequently,

$$\hat{I}_\nu \leqq I_\nu(w) \leqq I_o(\hat{u}_o) + \frac{\sigma}{2} + \nu \|w'\|^2_{\mathcal{U}^o} .$$

Hence, after adjusting $\nu = \nu(\sigma)$ so that $\nu(\sigma) \|w'\|^2_{\mathcal{U}^o} \leqq \frac{\sigma}{2}$, we get (3.9) and therefore

$$\limsup_{\nu \to 0} \hat{I}_\nu \leqq \hat{I}_o . \tag{3.11}$$

(3.11) together with (3.8) imply (3.7). By (3.11), for $\nu \leqq \nu(\sigma)$,

$$\|\hat{u}_\nu\|^2_{\mathcal{U}^o} + \nu\|\hat{u}'_\nu\|^2_{\mathcal{U}^o} \leqq \hat{I}_o \quad ,$$

thus, for a subsequence,

$$\hat{u}_\nu \to \hat{u} \qquad \text{weakly in } \mathcal{U}^o \ , \tag{3.12a}$$

$$\nu\hat{u}'_\nu \to 0 \qquad \text{strongly in } \mathcal{U}^o \ . \tag{3.12b}$$

To show that $\hat{u}$ is an optimal control for $(CP)_o$, observe that by Prop. 3.1,(i),(ii) ,

$$\Xi(\hat{u}_\nu) \to \Xi(\hat{u}) \quad \text{weakly in } L^2(0,T;V) \ , \quad \text{strongly in } L^2(Q) \ .$$

Hence, $\liminf\limits_{\nu\to 0} I_\nu(\hat{u}_\nu) \geqq I_o(\hat{u})$ . Simultaneously, by (3.11),

$$I_o(\hat{u}) \leqq \liminf_{\nu\to 0} I_\nu(\hat{u}_\nu) \leqq \hat{I}_o \quad ,$$

implying that $\hat{u}$ actually is optimal for $(CP)_o$ . Thus, the weak convergence in (3.5) and (3.6) have been shown. To complete the proof, it remains to show the strong convergence in (3.5). To this end, notice there is $\eta \geqq 0$ such that for a subsequence $\{\hat{u}_{\nu'}\}$ ,

$$\|\hat{u}_{\nu'}\|^2_{\mathcal{U}^o} \to \eta \quad \text{as} \quad \nu' \to 0 \ . \tag{3.13}$$

Due to Prop. 3.1,(i), by (3.12) and (3.13), we have

$$\liminf_{\nu'\to 0} I_{\nu'}(\hat{u}_{\nu'}) \geqq \|\Xi(\hat{u}) - \vartheta_d\|^2_{L^2(Q)} + \eta \ . \tag{3.14}$$

(3.14) together with (3.11) yield the inequality $\eta \leqq \|\hat{u}\|^2_{\mathcal{U}^o}$ . At the same time, by the weak l.s.c. of the norm,

$$\|\hat{u}\|^2_{\mathcal{U}^o} \leqq \liminf_{\nu'\to 0} \|\hat{u}_{\nu'}\|^2_{\mathcal{U}^o} = \eta \ .$$

Hence, $\lim\limits_{\nu'\to 0} \|\hat{u}_{\nu'}\|_{\mathcal{U}^o} = \|\hat{u}\|_{\mathcal{U}^o}$ . This gives (3.5) for the subsequence $\{\hat{u}_{\nu'}\}$ . It remains to show that (3.5) holds for the whole sequence $\{\hat{u}_\nu\}$ which satisfies (3.12a). To this end, observe that $\lim\limits_{\nu\to 0} \|\hat{u}_\nu\|_{\mathcal{U}^o} = \|\hat{u}\|_{\mathcal{U}^o}$ .

Suppose the converse, i.e., for some subsequence $\{\hat{u}_{\nu''}\}$,

$$\lim_{\nu''\to 0} \|\hat{u}_{\nu''}\|^2_{\mathcal{U}^o} = \bar{\eta} \neq \|\hat{u}\|^2_{\mathcal{U}^o} . \tag{3.15}$$

By repeating the arguments used for $\{\hat{u}_{\nu'}\}$, we can deduce though that $\bar{\eta} = \|\hat{u}\|^2_{\mathcal{U}^o}$. This contradicts (3.15), hence the proof is complete. □

## 4. CHARACTERIZATION OF OPTIMAL SOLUTIONS

### 4.1. Regularized control problem

We are going to exploit gradient-type algorithms for numerical solving the control problems, therefore differentiability of the state observation mapping and cost functional become of primary importance.

The control problems under study exhibit structural nonsmoothness due to lack of a sufficient regularity of the solution to variational inequality (VI). In order to ensure differentiability of the state observation mapping, with the differential explicitly given, we apply the regularization of (VI) as exposed in Sec. 2.2 . Consequently, after constructing discretizations to the regularized control problem, optimization techniques of gradient type can be applied for solving the problem numerically.

For $\mu, \varepsilon \in [0,1]$, let $\Xi^{\mu}_{\varepsilon} : \mathcal{U} \to L^2(Q)$ be defined by $\Xi^{\mu}_{\varepsilon}(u) = y'_{\mu\varepsilon}$, where $y_{\mu\varepsilon}$ is the solution of $(VI)^{\mu}_{\varepsilon}$ .

It is to be recalled here that the regularized variational inequality (VI) comprehends both $\bar{\rho}_{\mu} > 0$ and $\varepsilon > 0$ . Then the regularized counterpart of control problem (CP) assumes the form

<u>$(CP)^{\mu}_{\varepsilon}$ , $\mu \in [0,1]$ , $\varepsilon \in (0,1]$ .</u> $\inf_{u \in \mathcal{U}} \{ I^{\mu}_{\varepsilon}(u) = J(\Xi^{\mu}_{\varepsilon}(u),u) \}$ .

As up to now, at any of the parameters $\mu, \varepsilon$ vanishing, we skip index "0" in all relevant notations.

By the same arguments as in the proof of Prop. 3.1 it follows that, provided $\bar{\rho}_{\mu} > 0$ ,

$\Xi^{\mu}_{\varepsilon}$ is continuous from $\mathcal{U}$(weak) into $L^2(0,T;V)$(weak), compact from $\mathcal{U}^o$ into $L^2(Q)$,

Lipschitz continuous from $\mathcal{U}^o$ into $L^2(Q)$ ,
with the Lipschitz constant independent of $\varepsilon$ .(4.1)

Clearly, problem $(CP)^{\mu}_{\varepsilon}$ has at least one optimal solution $\hat{u}_{\mu\varepsilon} \in \mathcal{U}$.

The regularized state observation mapping $\Xi^{\mu}_{\varepsilon}$ is differentiable in the following sense.

PROPOSITION 4.1. Assume that $\bar{\rho}_{\mu}$ , $\varepsilon > 0$ . Then $\Xi^{\mu}_{\varepsilon}$ is Gateaux differentiable in $\mathcal{U}^o$ . Its Gateaux differential $D\Xi^{\mu}_{\varepsilon}$ is characterized by

$$D\Xi^{\mu}_{\varepsilon}(u)\, v \;=\; \xi'_{\mu\varepsilon} \quad , \qquad \forall\; u,v \in \mathcal{U}^o \quad , \tag{4.2}$$

where $\xi_{\mu\varepsilon}$ is the unique solution of the problem

$$\begin{cases} (\,D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon}(t))\,\xi'_{\mu\varepsilon}(t)\,,\,z\,) \;+\; a(\,\xi_{\mu\varepsilon}(t),z) \;= \\ \qquad = \;(\,\int_0^t v(s)\,ds\,,\,z\,)_{\Gamma} \;, \qquad \forall\; z \in V \;, \quad \text{a.a. } t \in [0,T], & (4.3a) \\ \xi_{\mu\varepsilon}(0) \;=\; 0 \qquad \text{in } \Omega \;, & (4.3b) \end{cases}$$

with $y'_{\mu\varepsilon} = \Xi^{\mu}_{\varepsilon}(u)$ , and $D\gamma_{\mu\varepsilon}(\cdot)$ being the Gateaux differential of $\gamma_{\mu\varepsilon} : H \to H$ .

<u>Proof.</u> For $\lambda > 0$ , denote $y'_{\lambda\mu\varepsilon} = \Xi^{\mu}_{\varepsilon}(u+\lambda v)$ ,
$\xi_{\lambda\mu\varepsilon} = (y_{\lambda\mu\varepsilon} - y_{\mu\varepsilon})/\lambda$ , $\quad \eta_{\lambda\mu\varepsilon} = (\,\gamma_{\mu\varepsilon}(y'_{\lambda\mu\varepsilon}) - \gamma_{\mu\varepsilon}(y'_{\mu\varepsilon}))/\lambda$ .
Observe that $\xi_{\lambda\mu\varepsilon}$ satisfies the system

$$\begin{cases} (\,\eta_{\lambda\mu\varepsilon}(t),z) \;+\; a(\,\xi_{\lambda\mu\varepsilon}(t),z) \;=\; (\,\int_0^t v(s)\,ds\,,\,z)_{\Gamma} \;, \\ \qquad\qquad\qquad \forall\; z \in V \;, \quad \text{a.a. } t \in [0,T] \;, & (4.4a) \\ \xi_{\lambda\mu\varepsilon}(0) \;=\; 0 \qquad \text{in } \Omega \;. & (4.4b) \end{cases}$$

Let us set $z = \xi'_{\lambda\mu\varepsilon}(t)$ in (4.4a) and integrate it over $[0,t]$, with $0 < t \leqq T$ . After integrating the right-hand side of the resulting inequality by parts, and applying Young's and Gronwall's inequalities, by the strict monotonicity of $\gamma_{\mu\varepsilon}$ , we get

$$\|\xi_{\lambda\mu\varepsilon}\|_{L^\infty(0,T;V)} + \bar{\rho}_\mu^{1/2}\|\xi'_{\lambda\mu\varepsilon}\|_{L^2(Q)} \leqq C'\|v\|_{L^2(\Sigma)} \leqq C\ , \tag{4.5a}$$

with a constant $C$ independent of $\lambda, \mu, \varepsilon$. Moreover, by (2.10),

$$\|\eta_{\lambda\mu\varepsilon}\|_{L^2(Q)} \leqq \frac{C}{\varepsilon}\|\xi'_{\lambda\mu\varepsilon}\|_{L^2(Q)} \leqq \frac{C}{\varepsilon\,\bar{\rho}_\mu^{1/2}}\ , \tag{4.5b}$$

with a constant $C$ independent of $\lambda, \mu, \varepsilon$. By (4.5), as $\lambda \to 0$,

$$\xi_{\lambda\mu\varepsilon} \to \xi_{\mu\varepsilon} \quad \text{weakly-* in } L^\infty(0,T;V), \quad \text{weakly in } H^1(0,T;H),$$

$$\eta_{\lambda\mu\varepsilon} \to \eta_{\mu\varepsilon} \quad \text{weakly in } L^2(Q)\ . \tag{4.6}$$

Hence, after passing in (4.4) to the limit as $\lambda \to 0$, we can see that equality (4.3a) is satisfied, with $\eta_{\mu\varepsilon}$ replacing $D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon})\,\xi'_{\mu\varepsilon}$. Besides, due to (4.4b), (4.3b) holds as well. Notice also that relation (4.2) follows directly by definition of the Gateaux differential. Hence, to complete the proof, it only remains to show that

$$\eta_{\mu\varepsilon} = D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon})\,\xi'_{\mu\varepsilon}\ . \tag{4.7}$$

By the mean-value theorem,

$$\eta_{\lambda\mu\varepsilon} = D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon})\,\xi'_{\lambda\mu\varepsilon} + \frac{1}{2}\lambda D^2\gamma_{\mu\varepsilon}(\bar{y}'_{\mu\varepsilon})(\xi'_{\lambda\mu\varepsilon})^2\ , \tag{4.8}$$

where $\bar{y}_{\mu\varepsilon} = (1-\beta)\,y_{\mu\varepsilon} + \beta\,y_{\lambda\mu\varepsilon}$, with some $\beta \in [0,1]$. Simultaneously, by (2.10) and (4.5) we have

$$\|\lambda D^2\gamma_{\mu\varepsilon}(\bar{y}'_{\mu\varepsilon})(\xi'_{\lambda\mu\varepsilon})^2\|_{L^1(Q)} \leqq \frac{C|\lambda|}{\varepsilon^2}\|\xi'_{\lambda\mu\varepsilon}\|^2_{L^2(Q)} \leqq \frac{C\,|\lambda|}{\varepsilon^2\,\bar{\rho}_\mu}\ ,$$

where $C$ is a positive constant independent of $\lambda, \mu, \varepsilon$. Hence, as $\lambda \to 0$,

$$\lambda D^2\gamma_{\mu\varepsilon}(\bar{y}'_{\mu\varepsilon})(\xi'_{\lambda\mu\varepsilon})^2 \to 0 \quad \text{strongly in } L^1(Q)\ . \tag{4.9}$$

According to (4.8) and (4.9), by (4.6) we get relation (4.7). This completes the proof. □

Let us notice that problem (4.3) has the unique solution $\xi_{\mu\varepsilon} \in L^\infty(0,T;V) \cap H^1(0,T;H)$ which satisfies estimate (4.5a).

To study convergence of regularized control problems, we need some estimates of the errors due to regularization of variational inequality (VI). We recall here those estimates, expressed in terms of the state observation mapping $\Xi$ (cf., [9] ).

For any $u \in \mathcal{U}^o$ , the error due to parabolic regularization admits the estimate

$$\bar{\rho}_\mu^{1/2} \, \| \Xi(u) - \Xi^\mu(u) \|_{L^2(Q)} \leqq C_o \, \mu^{1/2} \quad , \tag{4.10}$$

with constant $C_o$ defined as in (2.15). In the parabolic case ($\bar{\rho} > 0$), under the additional hypothesis on initial data about the phase transition point,

(A6) $\operatorname{mes} \{ x \in \Omega \mid 0 < \vartheta_o(x) < \varepsilon \} \leqq C \varepsilon$ , with $C \neq C(\varepsilon)$ ,

for any $u \in \mathcal{U}^o$ the relevant estimate on the error due to smoothing is

$$\bar{\rho} \, \| \Xi(u) - \Xi_\varepsilon(u) \|_{L^2(Q)} \leqq C \, \varepsilon^{1/2} \quad , \tag{4.11}$$

where $C$ is a positive constant independent of $\varepsilon$ .

## 4.2. Necessary conditions of optimality

Let us consider the regularized problem $(CP)^\mu_\varepsilon$ . We define the adjoint state as solution of the problem

$(AP)^\mu_\varepsilon$ .
$$\begin{cases} ( D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon}(t)) \, p'_{\mu\varepsilon}(t) \, , z ) - a(p_{\mu\varepsilon}(t), z) = \\ \qquad = (y'_{\mu\varepsilon}(t) - \vartheta_d(t), z) \; , \quad \forall \, z \in V \, , \quad \text{a.a. } t \in [0,T] \, , & (4.12a) \\ p_{\mu\varepsilon}(T) = 0 \quad \text{in } \Omega \; , & (4.12b) \end{cases}$$

where $y'_{\mu\varepsilon} = \Xi^\mu_\varepsilon(u)$ .

Problem (4.12) has a unique solution $p_{\mu\varepsilon} \in L^\infty(0,T;V) \cap$ $\cap \, H^1(0,T;H)$ . Optimal solutions to $(CP)^\mu_\varepsilon$ can be given the following characterization.

PROPOSITION 4.2. Assume that $\bar{\rho}_\mu, \varepsilon > 0$ . Let $\hat{u}_{\mu\varepsilon} \in \mathcal{U}$ be an arbitrary optimal control to $(CP)^\mu_\varepsilon$ and $\hat{y}'_{\mu\varepsilon} = \Xi^\mu_\varepsilon(\hat{u}_{\mu\varepsilon})$ represent the corresponding optimal state. Then there exists a function

$\hat{p}_{\mu\varepsilon} \in L^{\infty}(0,T;V) \cap H^1(0,T;H)$ which satisfies $(AP)^{\mu}_{\varepsilon}$ corresponding to $\hat{y}'_{\mu\varepsilon}$, and such that

$$\hat{p}_{\mu\varepsilon}|_{\Sigma} = \alpha\, \hat{u}_{\mu\varepsilon} \qquad \text{if} \quad \mathcal{U} = \mathcal{U}^{o} , \tag{4.13}$$

whereas

$$\hat{\hat{p}}_{\mu\varepsilon}|_{\Sigma} = \alpha\, \hat{u}'_{\mu\varepsilon} , \quad \hat{\hat{p}}_{\mu\varepsilon}(0)|_{\Gamma} = \alpha\, \hat{u}_{\mu\varepsilon}(0) \quad \text{if} \quad \mathcal{U} = \mathcal{U}^{1} , \tag{4.14}$$

where
$$\hat{\hat{p}}_{\mu\varepsilon}(t) = \int_t^T \hat{p}_{\mu\varepsilon}(s)\, ds .$$

Proof. If $\mathcal{U} = \mathcal{U}^{o}$, then the Gateaux differential of $I^{\mu}_{\varepsilon}$ is characterized by

$$DI^{\mu}_{\varepsilon}(u)\, v = (\Xi^{\mu}_{\varepsilon}(u) - \vartheta_d , D\Xi^{\mu}_{\varepsilon}(u)\, v)_{L^2(Q)} + \alpha\, (u,v)_{L^2(\Sigma)} ,$$
$$\forall\, u,v \in \mathcal{U}^{o} . \tag{4.15}$$

But according to (4.2),(4.3) and (4.12), we have

$$(\Xi^{\mu}_{\varepsilon}(u) - \vartheta_d , D\Xi^{\mu}_{\varepsilon}(u)\, v)_{L^2(Q)} = (\Xi^{\mu}_{\varepsilon}(u) - \vartheta_d , \xi'_{\mu\varepsilon})_{L^2(Q)} =$$
$$= \int_0^T [(D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon}(t))\, p'_{\mu\varepsilon}(t) , \xi'_{\mu\varepsilon}(t)) - a(p_{\mu\varepsilon}(t), \xi'_{\mu\varepsilon}(t))]dt =$$
$$= \int_0^T [(D\gamma_{\mu\varepsilon}(y'_{\mu\varepsilon}(t))\, \xi'_{\mu\varepsilon}(t), p'_{\mu\varepsilon}(t)) + a(\xi_{\mu\varepsilon}(t), p'_{\mu\varepsilon}(t))]dt =$$
$$= \int_0^T \left( \int_0^t v(s)ds , p'_{\mu\varepsilon}(t)\right)_{\Gamma} dt = -(p_{\mu\varepsilon}, v)_{L^2(\Sigma)} . \tag{4.16}$$

In view of (4.15) and (4.16), the optimality condition $DI^{\mu}_{\varepsilon}(\hat{u}_{\mu\varepsilon}) = 0$ implies directly (4.13). If $\mathcal{U} = \mathcal{U}^{1}$ then due to (4.16), after an appropriate integration by parts, we get

$$DI^{\mu}_{\varepsilon}(u)v = (\Xi^{\mu}_{\varepsilon}(u) - \vartheta_d , D\Xi^{\mu}_{\varepsilon}(u)v)_{L^2(Q)} + \alpha\, (u',v')_{L^2(\Sigma)} +$$
$$+ \alpha(u(0),v(0))_{\Gamma} = (\tilde{p}'_{\mu\varepsilon}, v)_{L^2(\Sigma)} + \alpha\, (u',v')_{L^2(\Sigma)} +$$
$$+ \alpha\, (u(0),v(0))_{\Gamma} = (-\tilde{p}_{\mu\varepsilon} + \alpha\, u' , v')_{L^2(\Sigma)} +$$
$$+ (-\tilde{p}_{\mu\varepsilon}(0) + \alpha\, u(0) , v(0))_{\Gamma} , \quad \forall\, u,v \in \mathcal{U}^{1} .$$

Hence, relations (4.14) follow. □

## 4.3. Convergence of regularized control problems

We shall show that the regularization approach presented above is correct, i.e., the regularized control problems in a certain sense approximate the original one. To begin, we consider the parabolic situation, with regularization reduced to the smoothing, i.e., $\varepsilon > 0$ and $\mu = 0$.

THEOREM 4.1. Consider control problem (CP) in the parabolic case ($\bar{\rho} > 0$), with $\mathcal{U} = \mathcal{U}^{o}$ or $\mathcal{U}^{1}$. Assume that hypotheses (A1), $(A2)^{o}$, (A3)-(A6) hold. Let $\{\hat{u}_{\varepsilon}\} \subset \mathcal{U}$ be a sequence of optimal controls to problems $(CP)_{\varepsilon}$. Then, for a subsequence, as $\varepsilon \to 0$,

$$\hat{u}_{\varepsilon} \to \hat{u} \quad \text{strongly in } \mathcal{U}, \tag{4.17}$$

$$\Xi_{\varepsilon}(\hat{u}_{\varepsilon}) \to \Xi(\hat{u}) \quad \begin{array}{l}\text{weakly in } L^{2}(0,T;V) \text{ and} \\ \text{strongly in } L^{2}(Q) \text{ if } \mathcal{U} = \mathcal{U}^{o},\end{array} \tag{4.18a}$$

$$\begin{array}{l}\text{weakly-* in } L^{\infty}(0,T;V) \text{ and} \\ \text{weakly in } H^{1}(Q) \text{ if } \mathcal{U} = \mathcal{U}^{1},\end{array} \tag{4.18b}$$

$$\hat{I}_{\varepsilon} \to \hat{I}, \quad I(\hat{u}_{\varepsilon}) \to \hat{I} \quad \text{with convergence rate } O(\varepsilon^{1/2}), \tag{4.19}$$

where $\hat{u}$ is an optimal control for (CP); $\hat{I}_{\varepsilon} = I_{\varepsilon}(\hat{u}_{\varepsilon})$, $\hat{I} = I(\hat{u})$.

<u>Proof.</u> First we show that $\{\hat{u}_{\varepsilon}\}$ is bounded in $\mathcal{U}$. Indeed, since

$$I_{\varepsilon}(\hat{u}_{\varepsilon}) = J(\Xi_{\varepsilon}(\hat{u}_{\varepsilon}),\hat{u}_{\varepsilon}) \leqq J(\Xi_{\varepsilon}(\hat{u}),\hat{u}), \tag{4.20}$$

by (4.11) we have

$$\limsup_{\varepsilon \to 0} I_{\varepsilon}(\hat{u}_{\varepsilon}) \leqq J(\Xi(\hat{u}),\hat{u}) = \hat{I}. \tag{4.21}$$

Thus, $\|\hat{u}_{\varepsilon}\|_{\mathcal{U}} \leqq C$ with $C \neq C(\varepsilon)$, and, for a subsequence, $\hat{u}_{\varepsilon} \to \hat{u}$ weakly in $\mathcal{U}$ as $\varepsilon \to 0$. Due to (2.14a,b), we get the corresponding uniform bounds for $\{\hat{y}_{\varepsilon}\}$. Hence, at $\mathcal{U} = \mathcal{U}^{o}$,

$$\hat{y}_{\varepsilon} \to \hat{y} \quad \begin{array}{l}\text{weakly-* in } W^{1,\infty}(0,T;H), \\ \text{weakly in } H^{1}(0,T;V);\end{array} \tag{4.22a}$$

at $\mathcal{U} = \mathcal{U}^{1}$,

$$\hat{y}_\varepsilon \to \hat{y} \quad \text{weakly-* in } W^{1,\infty}(0,T;V) , \quad \text{weakly in } H^2(0,T;H) . \tag{4.22b}$$

To show that $\hat{y}' = \Xi(\hat{u})$ , we pass to the limit with $\varepsilon \to 0$ in $(VI)_\varepsilon$ , written in the form (2.8) ($\mu = 0$), with $F$, $f_o$ to be replaced by $F_\varepsilon$ and $f_{o\varepsilon}$ , respectively. As in Prop. 3.1(i), we conclude that $\hat{y}$ satisfies (VI) with $\hat{u}$ . In addition, notice that since

$$\| \Xi_\varepsilon(\hat{u}_\varepsilon) - \Xi(\hat{u}) \|_{L^2(Q)} \leqq \| \Xi_\varepsilon(\hat{u}_\varepsilon) - \Xi(\hat{u}_\varepsilon) \|_{L^2(Q)} + \\ + \| \Xi(\hat{u}_\varepsilon) - \Xi(\hat{u}) \|_{L^2(Q)} ,$$

due to (4.11) and Prop. 3.1,(ii), we have $\Xi_\varepsilon(\hat{u}_\varepsilon) \to \Xi(\hat{u})$ strongly in $L^2(Q)$ as $\varepsilon \to 0$ . By weak l.s.c. of the norm,

$$J(\Xi(\hat{u}),\hat{u}) \leqq \liminf_{\varepsilon \to 0} J(\Xi_\varepsilon(\hat{u}_\varepsilon),\hat{u}_\varepsilon) . \tag{4.23}$$

Simultaneously, by (4.21) we get inequality $J(\Xi(\hat{u}),\hat{u})) \leqq \hat{I}$ . This implies $\hat{u}$ is an optimal control for (CP). Assertion (4.18) has been shown. Clearly, by (4.21) and (4.23), $\hat{I}_\varepsilon \to \hat{I}$ as $\varepsilon \to 0$ .

To conclude (4.17), it remains to notice that the strong convergence of $\{\hat{u}_\varepsilon\}$ in $\mathcal{U}$ follows by (4.18) in the same way as in Thm. 3.2.

Finally, to show (4.19), observe that due to (4.11) and by a priori bounds (2.14a) on $\Xi_\varepsilon(\hat{u})$ and $\Xi(\hat{u})$ ,

$$| \hat{I} - I_\varepsilon(\hat{u}) | \leqq C \| \Xi(\hat{u}) - \Xi_\varepsilon(\hat{u}) \|_{L^2(Q)} \leqq C\, \varepsilon^{1/2} , \tag{4.24}$$

with a positive constant $C$ independent of $\varepsilon$ . Similarly,

$$| I(\hat{u}_\varepsilon) - \hat{I}_\varepsilon | \leqq C\, \varepsilon^{1/2} , \tag{4.25}$$

with $C$ independent of $\varepsilon$ , because of the uniform boundedness of $\{\hat{u}_\varepsilon\}$ in $\mathcal{U}$ . By (4.24),(4.25) and since $\hat{I} \leqq I(\hat{u}_\varepsilon)$ , $\hat{I}_\varepsilon \leqq$ $\leqq I_\varepsilon(\hat{u})$ , we have

$$0 \leqq I(\hat{u}_\varepsilon) - \hat{I} \leqq |I(\hat{u}_\varepsilon) - \hat{I}_\varepsilon| + |I_\varepsilon(\hat{u}) - \hat{I}| \leqq C\, \varepsilon^{1/2} , \tag{4.26}$$

with $C$ independent of $\varepsilon$ . Eventually, according to (4.25) and (4.26),

$$| \hat{I}_\varepsilon - \hat{I} | \leq | \hat{I}_\varepsilon - I(\hat{u}_\varepsilon) | + | I(\hat{u}_\varepsilon) - \hat{I} | \leq C\,\varepsilon^{1/2} \ .$$

With the last estimate, the proof is complete. □

In the degenerate case ($\bar{\rho} = 0$), there is a two-step regularization, with two positive parameters $\mu, \varepsilon$. Thus, it is of interest then to consider both iterative and joint convergences of the solutions to the regularized control problems $(CP)^\mu_\varepsilon$ with respect to $\mu$ and $\varepsilon$.

THEOREM 4.2. Consider problem $(CP)^\mu_\varepsilon$, $\mu, \varepsilon \in (0,1]$, in the degenerate case ($\bar{\rho} = 0$), with $\mathcal{U} = \mathcal{U}^0$ or $\mathcal{U}^1$. Assume that (A1), $(A2)^0$, (A3)-(A6) hold. Let $\{\hat{u}_{\mu\varepsilon}\} \subset \mathcal{U}$ be a sequence of optimal controls for $(CP)^\mu_\varepsilon$.

(I) Iterative convergence: $\varepsilon \to 0$, $\mu \to 0$.

(i) Assume $\mu > 0$ fixed. Then there exists a subsequence of $\{\hat{u}_{\mu\varepsilon}\}$, such that $\hat{u}_{\mu\varepsilon} \to \hat{u}_\mu$ strongly in $\mathcal{U}$ as $\varepsilon \to 0$, assertions (4.18) and (4.19) on the convergences of the optimal states $\Xi^\mu_\varepsilon(\hat{u}_{\mu\varepsilon}) \to \Xi^\mu(\hat{u}_\mu)$, and the minimal values of the cost functionals, $\hat{I}^\mu_\varepsilon \to \hat{I}^\mu$, $I^\mu(\hat{u}_{\mu\varepsilon}) \to \hat{I}^\mu$ take place, with $\hat{u}_\mu$ representing an optimal control for problem $(CP)^\mu$, $\hat{I}^\mu = I^\mu(\hat{u}_\mu)$.

(ii) Let $\{\hat{u}_\mu\}$ be a sequence of optimal controls for $(CP)^\mu$. Then there exists a subsequence of $\{\hat{u}_\mu\}$, such that as $\mu \to 0$,

$$\hat{u}_\mu \to \hat{u} \quad \text{strongly in } \mathcal{U} , \tag{4.27}$$

$$\Xi^\mu(\hat{u}_\mu) \to \Xi(\hat{u}) \quad \text{weakly in } L^2(0,T;V) , \tag{4.28}$$

$$\hat{I}^\mu \to \hat{I} , \tag{4.29}$$

where $\hat{u}$ is an optimal control for problem (CP); $\hat{I} = I(\hat{u})$.

(II) Joint convergence: $\mu, \varepsilon \to 0$. Assume $\varepsilon \leq \varepsilon_0\, \mu^{2+\delta}$, $\varepsilon_0, \delta > 0$. Then, for a subsequence of $\{\hat{u}_{\mu\varepsilon}\}$, as $\mu, \varepsilon \to 0$, the assertions (4.27)-(4.29) hold for $\hat{u}_{\mu\varepsilon}$, $\Xi^\mu_\varepsilon(\hat{u}_{\mu\varepsilon})$ and $\hat{I}^\mu_\varepsilon$, respectively.

Proof. (I) Assertion (i) follows immediately from Thm. 4.1. In the proof of (ii) we make use of convergences (2.22) and, moreover, the following property:

if $u_\mu \to u$ weakly in $\mathcal{U}$ as $\mu \to 0$, then

$$\Xi^\mu(u_\mu) \to \Xi(u) \quad \text{weakly in } L^2(0,T;V) . \tag{4.30}$$

To show (4.30), observe that since $\{u_\mu\}$ is bounded in $\mathcal{U}$, a priori estimates (2.14a) imply uniform bounds on $\{y_\mu\}$ in $H^1(0,T;V)$, with $y'_\mu = \Xi^\mu(u_\mu)$. Therefore, for a subsequence, $y_\mu \to y$ weakly in $H^1(0,T;V)$ as $\mu \to 0$. In order to prove that $y' = \Xi(u)$, we pass to the limit as $\mu \to 0$ in $(VI)^\mu$, given the form (2.8). Then, as in Prop. 3.1,(i), due to (2.3) and (2.7), (4.30) follows. On account of (2.22) and (4.30), we can argue as in the proof of Thm. 4.1, to show assertions (4.27)-(4.29).

(II) The following convergences hold: for any $u \in \mathcal{U}$,

$$\Xi^\mu_\varepsilon(u) \to \Xi(u) \quad \text{weakly in } L^2(0,T;V) \quad \text{as } \mu,\varepsilon \to 0 ; \tag{4.31a}$$

if $u_{\mu\varepsilon} \to u$ weakly in $\mathcal{U}$ as $\mu,\varepsilon \to 0$, then

$$\Xi^\mu_\varepsilon(u_{\mu\varepsilon}) \to \Xi(u) \quad \text{weakly in } L^2(0,T;V) . \tag{4.31b}$$

By the uniform estimate (2.14a) on $y_{\mu\varepsilon}$,

$$y_{\mu\varepsilon} \to y \quad \text{weakly in } H^1(0,T;V) \quad \text{as } \mu,\varepsilon \to 0 .$$

To show that $y' = \Xi(u)$, we pass to the limit with $\mu,\varepsilon \to 0$ in $(VI)^\mu_\varepsilon$ written in the form (2.8), with $F^\mu$, $f_\mu$ to be replaced by $F^\mu_\varepsilon$ and $f_{\mu\varepsilon}$, respectively. Again, we follow the arguments of Prop. 3.1,(i), this time exploiting (2.7) at $\mu,\varepsilon \to 0$ and the convergence

$$f_{\mu\varepsilon} \to f_o \quad \text{strongly in } L^2(Q) \quad \text{as } \mu,\varepsilon \to 0$$

(a consequence of (2.3),(2.11b) ), to conclude that $y$ satisfies (VI) with $u$. This yields (4.31b) and, similarly, (4.31a). Now, we shall show that for any $u \in \mathcal{U}$,

$$\Xi^\mu_\varepsilon(u) \to \Xi(u) \quad \text{strongly in } L^2(Q) \quad \text{as } \mu,\varepsilon \to 0 , \tag{4.31c}$$

provided an appropriate relation between $\mu$ and $\varepsilon$. Indeed, notice that

$$\|\Xi^\mu_\varepsilon(u) - \Xi(u)\|_{L^2(Q)} \leqq \|\Xi^\mu_\varepsilon(u) - \Xi^\mu(u)\|_{L^2(Q)} + \|\Xi^\mu(u) - \Xi(u)\|_{L^2(Q)} . \tag{4.32}$$

By estimate (4.11), with $\bar{\rho}$ to be replaced by $\mu$ (since the problem is parabolically regularized), for every $u \in \mathcal{U}$

$$\|\Xi^{\mu}(u) - \Xi^{\mu}_{\varepsilon}(u)\|_{L^2(Q)} \leqq C \frac{\varepsilon^{1/2}}{\mu} \leqq C \varepsilon_0^{1/2} \mu^{\delta/2} , \qquad (4.33)$$

with a constant $C$ independent of $\mu, \varepsilon$ and $u$. On account of (2.22b) and (4.33), estimate (4.32) implies (4.31c).

Eventually, with the properties (4.31,a-c) for $\Xi^{\mu}_{\varepsilon}$, we can again apply the same arguments as in Thm. 4.1, to conclude assertion (II). □

Remark. How to construct necessary optimality conditions for control problem (CP) in an explicit form, remains an open question. It is so due to lack of any global regularity of the free boundary. Were it at least of zero Lebesgue measure in $Q$ (what is unknown and rather questionable), a construction due to Tiba [14] would apply in the case of the parabolic two-phase Stefan problem.

## 5. CONCLUDING COMMENTS

Discrete approximations to optimal control problem (CP) are studied in the forthcoming paper [11]. In [11], the problem is discretized by employing linear finite elements in space and finite differences in time. The approximations equally comprise schemes with regularization of the state observation mapping and schemes without such regularization.

The regularization techniques exploited in this paper turn out useful for the numerical analysis of the corresponding discrete approximations. In view of a suitable regularity of solutions to the regularized problems $(VI)^{\mu}_{\varepsilon}$, and simultaneously due to the error estimates established for parabolic regularization and smoothing, an application of the regularization procedure brings estimates on the errors which characterize accuracy of the proposed discrete approximations. Moreover, as it has been shown in Sec. 4, regularization applied to the control problem assures differentiability of the state observation mapping. This, in turn, implies differentiability of the cost functional and gives rise to optimality conditions in an explicit form, therefore providing a constructive gradient-type algorithm for numerical solving the control problem under consideration.

REFERENCES


[1] V. Barbu, Optimal Control of Variational Inequalities, Pitman, Boston,1984.

[2] A. Bermudez, C. Saguez, Optimal control of variational inequalities; optimality conditions and numerical methods, in: A. Bossavit et al., Eds., Free Boundary Problems - - Applications and Theory, Vol. IV, Pitman, Boston, 1985, 478-487.

[3] A. Bermudez, C. Saguez, Optimal control of variational inequalities, Control & Cyber., 14 (1985), 9-30.

[4] M.G. Crandall, A. Pazy, Semigroups of nonlinear contractions and dissipative sets, J. Func. Anal., 3 (1969), 376-418.

[5] C.M. Elliott, J.R. Ockendon, Weak and Variational Methods for Moving Boundary Problems, Pitman, Boston, 1982.

[6] K.-H. Hoffmann, M. Niezgódka, J. Sprekels, Feedback control of multidimensional two-phase Stefan problems via thermostats, to appear.

[7] U. Hornung, A parabolic-elliptic variational inequality, Manuscripta Math., 39 (1982), 155-172.

[8] P. Neittaanmäki, D. Tiba, A finite element approximation of the boundary control of two-phase Stefan problems, Report No. 4, Lappeenranta Univ. Tech., 1983.

[9] I. Pawlow, A variational inequality approach to generalized two-phase Stefan problem in several space variables, Annali Matem. Pura Applicata, 131 (1982), 333-373.

[10] I. Pawlow, Optimal control of two-phase Stefan problems - numerical solutions, in: K.-H. Hoffmann,W. Krabs, Eds., Optimal Control of PDEs, Birkhäuser, Basel, in print.

[11] I. Pawlow, Discrete approximations to boundary control of parabolic-elliptic multiphase Stefan problems, to appear.

[12] L. Rubinstein, Stefan Problem, Zvaigzne, Riga, 1967 (in Russian)

[13] C. Saguez, Controle optimal de systèmes à frontière libre, Thèse, Univ. Technol., Compiegne, 1980.

[14] D. Tiba, Boundary control for a Stefan problem, in: K.-H. Hoffmann, W. Krabs, Eds., Optimal Control for PDEs, Birkhauser, Basel, 1984, 229-242.

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